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Solvability and Spectral Properties of Boundary Value Problems for Equations of Even Order

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ABSTRACT

We study boundary value problems for an equation of the order $2k$ and prove regular and strong solvability of it, investigate spectrum of the problem. In case of even k we obtain a priori estimate for the solution in the norm of the Sobolev space and prove solvability almost everywhere.

Keywords: solvability, boundary value problem, spectrum, a priori estimate, regular solvability, strong solvability, the Fourier series, the Cauchy-Schwarz inequality, the Bessel inequality, the Perceval equality, the Lipchitz condition, even, odd, almost everywhere solution.

INTRODUCTION

Boundary value problems for the equations of the 3rd and 4th order first were investigated by Hadamard,(1933) and Sjöstrand,(1937), and developed by Davis,(1954), Bitsadze,(1961), Salahitdinov,(1974), Dzhuraev,(1979), Wolfersdorf,(1969) and others.

Boundary value problems for the equations of the order 4 were studied by Dzhuraev and Sopuev,(2000), Salahitdinov and Amanov,(2005), Nicolescu,(1954), Roitman,(1971) and Sobolev,(1988).

In present paper we study boundary value problems for an equation of the order $2k$.

Statement of the Problems

We consider the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}} - \frac{\partial^2 u}{\partial t^2} = f(x,t)\,, \tag{1}$$

in the domain $\Omega = \{\ (x,t):\ 0 < x < p,\ 0 < t < T\}$, where $k \geq 2$ is fixed positive integer.

Problem 1

Find the solution $u(x,t)$ of the equation (1) in the domain Ω satisfying conditions

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t) = \frac{\partial^{2m}u}{\partial x^{2m}}(p,t) = 0\,, \quad m = 0,1,\ldots,k-1\,, \quad 0 \leq t \leq T\,, \tag{2}$$

$$u(x,0) = 0,\ \ u(x,T) = 0, \qquad 0 \leq x \leq p\,. \tag{3}$$

Problem 2

Find the solution $u(x,t)$ of the equation (1) in the domain Ω satisfying conditions (3) and

$$\frac{\partial^{2m+1}u}{\partial x^{2m+1}}(0,t) = \frac{\partial^{2m+1}u}{\partial x^{2m+1}}(p,t) = 0\,, \ m = 0,1,\ldots,k-1\,, \quad 0 \leq t \leq T\,, \tag{4}$$

Problem 3

Find the solution $u(x,t)$ of the equation (1) in the domain Ω satisfying conditions (2) and

$$u(x,0) = 0,\ \ u_t(x,T) = 0, \qquad 0 \leq x \leq p\,. \tag{5}$$

Problem 4

Find the solution $u(x,t)$ of the equation (1) in the domain Ω satisfying conditions (2) and

$$u(x,0) = u(x,T), \qquad u_t(x,0) = u_t(x,T), \qquad 0 \le x \le p. \tag{6}$$

We investigate Problem 1 in detail and other problems can be similarly examined.

Let

$$V(\Omega) = \left\{ u : u \in C_{x,t}^{2k-2,0}(\overline{\Omega}) \cap C_{x,t}^{2k,2}(\Omega), \text{ and conditions (2), (3) are true} \right\},$$

$$W_1(\Omega) = \{ f : f \in C_{x,t}^{1,0}(\overline{\Omega}), f(0,t) = f(p,t) = 0, \frac{\partial f}{\partial x} \in Lip_\alpha[0,p]$$

is uniformly in $t, 0 < \alpha \le 1 \}$,

$$W_2(\Omega) = \{\}\left\{ f : f \in C_{x,t}^{k,0}(\overline{\Omega}), \frac{\partial^{k+1} f}{\partial x^{k+1}} \in L_2(\Omega), \frac{\partial^{2m} f}{\partial x^{2m}} = 0, \quad with \quad m = 0,1,\dots,\frac{k-1}{2} \right\}$$

We define the operator L

$$Lu \equiv \left(\frac{\partial^{2k}}{\partial x^{2k}} - \frac{\partial^2}{\partial t^2} \right) u$$

mapping the domain $V(\Omega)$ into $C(\Omega)$.

Definition 1

A function $u(x,t) \in V(\Omega)$ is called the regular solution of the problem 1 with $f(x,t) \in C(\Omega)$ if it satisfies the equation (1) in the domain Ω.

Definition 2

A function $u(x,t) \in L_2(\Omega)$ is called the strong solution of the problem 1 with $f \in L_2(\Omega)$ if there exists a sequence $u_n \in V(\Omega)$, $n \in N$, such that $\|u_n - u\|_{L_2(\Omega)} \to 0$, $\|Lu_n - f\|_{L_2(\Omega)} \to 0$ as $n \to \infty$.

Denote by $\overset{0}{W}{}_2^{2k,2}(\Omega)$ the closure of the set $V(\Omega)$ in the norm

$$\|u\|^2_{W_2^{2k,2}(\Omega)} = \iint\limits_{\Omega} \left[\sum_{m=0}^{2k} \left(\frac{\partial^m u}{\partial x^m} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial^2 u}{\partial t^2} \right)^2 + \sum_{m=2}^{k+1} \left(\frac{\partial^m u}{\partial t \partial x^{m-1}} \right)^2 \right] dxdt$$

and by $\overset{0}{W}{}_2^{k,1}(\Omega)$ the closure of the set $V(\Omega)$ in the norm

$$\|u\|^2_{W_2^{k,1}(\Omega)} = \iint\limits_{\Omega} \left[\sum_{m=0}^{k} \left(\frac{\partial^m u}{\partial x^m} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dxdt .$$

It is clear that $\overset{0}{W}{}_2^{2k,2}(\Omega)$ and $\overset{0}{W}{}_2^{k,1}(\Omega)$ are subspaces of the Sobolev spaces $W_2^{2k,2}(\Omega)$ and $W_2^{k,1}(\Omega)$ respectively. If we complete the set $V(\Omega)$, then operator L is also completed. Let $\overline{L}$ be the closure of operator L in both cases with $D(\overline{L}) = \overset{0}{W}{}_2^{2k,2}(\Omega)$ if k is even, and $D(\overline{L}) = \overset{0}{W}{}_2^{k,1}$ if k is odd.

A Priori Estimate

It is true the following

Lemma 1. Let $u(x,t)$ be a regular solution of Problem 1 having continuous derivatives

$$\frac{\partial^{m+1} u}{\partial x^m \partial t}(0,t),\ \frac{\partial^{2k-1} u}{\partial x^{2k-1}},\ \frac{\partial^{2k} u}{\partial x^{2k}},\ \frac{\partial u}{\partial t},\ \frac{\partial^2 u}{\partial t^2}, \qquad m = 0,1,...,k ,$$

in Ω and belonging to $L_2(\Omega), f(x,t) \in C(\Omega) \cap L_2(\Omega)$, where k is odd. Then there exists a constant $C > 0$ that depends only on sizes of the domain and the number k and doesn't depend on the function $u(x,t)$ such that

$$\|u\|_{W_2^{2k,2}(\Omega)} \le C \|f\|_{L_2(\Omega)}. \tag{7}$$

Proof. We multiply by $u(x,t)$ both sides of the equation (1) and integrate it over the region Ω to obtain

$$\iint_\Omega u\left(\frac{\partial^{2k}u}{\partial x^{2k}} - \frac{\partial^2 u}{\partial t^2}\right) dxdt = \iint_\Omega u \ f \ dxdt . \tag{8}$$

Using the formulas

$$u\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial t}\right) - \left(\frac{\partial u}{\partial t}\right)^2,$$

$$u\frac{\partial^2 ku}{\partial x^{2k}} = \sum_{m=0}^{k-1}(-1)^m \frac{\partial}{\partial x}\left(\frac{\partial^m u}{\partial x^m}\cdot\frac{\partial^{2k-1-m}}{\partial x^{2k-m-1}}\right) + (-1)^k\left(\frac{\partial^k u}{\partial x^k}\right)^2,$$

and conditions (2), (3), the equation (8) becomes

$$\left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(\Omega)} = \iint_\Omega u \ f \ dxdt . \tag{9}$$

Applying the following evident inequality

$$|ab| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2$$

with arbitrary $\varepsilon > 0$ to the right-hand side of (9) we obtain

$$\left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(\Omega)} \le \frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L_2(\Omega)}. \tag{10}$$

It is obvious that

$$u^2(x,t)=\int_0^t \frac{\partial}{\partial\tau}\left(u^2(x,\tau)\right)\,d\tau=2\int_0^t u(x,\tau)\frac{\partial u}{\partial\tau}\,d\tau\le 2\int_0^t \left|u(x,\tau)\frac{\partial u}{\partial\tau}\right|\,d\tau .$$

Integrating it with respect to x from 0 to p gives

$$\int_0^p u^2(x,t)\,dx\le\ 2\int_0^p\int_0^T \left|u(x,t)\frac{\partial u}{\partial t}\right|\,dtdx .$$

Applying the Cauchy-Schwarz inequality to the right-hand side we have

$$\int_0^p u^2(x,t)\,dx\le\ 2\|u\|_{L_2(\Omega)}\cdot\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)} .$$

Integrating it with respect to t from 0 to T yields

$$\|u\|^2_{L_2(\Omega)}\ \le 2T\|u\|_{L_2(\Omega)}\cdot\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)} .$$

Dividing by $\|u\|_{L_2(\Omega)}$ both parts of this inequality and squaring it we obtain from (10)

$$\|u\|^2_{L_2(\Omega)}\ \le 2T^2\varepsilon\|u\|^2_{L_2(\Omega)}+\frac{2T^2}{\varepsilon}\|f\|^2_{L_2(\Omega)} . \qquad (11)$$

If we add the inequalities (10) and (11) by choosing $\varepsilon=\dfrac{1}{4T^2+1}$ and multiply by 2 both sides of it and replace coefficients 2 by 1 on the left-hand side, then we obtain

$$\|u\|^2_{L_2(\Omega)}\ +\left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)}\le\left(4T^2+1\right)^2\|f\|^2_{L_2(\Omega)} . \qquad (12)$$

If we square both parts of (1) and integrate over Ω, then we have

$$\left\|\frac{\partial^{2k}u}{\partial x^{2k}}\right\|^2_{L_2(\Omega)} - 2\iint_\Omega \frac{\partial^2 u}{\partial t^2}\ \frac{\partial^{2k}u}{\partial x^{2k}}\ dxdt + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2(\Omega)} = \|f\|^2_{L_2(\Omega)}. \qquad (13)$$

Let us rearrange the integrand by the following way

$$\frac{\partial^2 u}{\partial t^2}\cdot\frac{\partial^{2k}u}{\partial x^{2k}} = (-1)^0\frac{\partial}{\partial x}\left(\frac{\partial^{2+0}u}{\partial t^2}\cdot\frac{\partial^{2k-1}u}{\partial x^{2k-1}}\right) + (-1)\frac{\partial^{2+1}u}{\partial t^2\partial x}\cdot\frac{\partial^{2k-1}u}{\partial x^{2k-1}} =$$

$$= (-1)^0\frac{\partial}{\partial x}\left(\frac{\partial^{2+0}u}{\partial t^2}\cdot\frac{\partial^{2k-1}u}{\partial x^{2k-1}}\right) + (-1)^1\frac{\partial}{\partial x}\left(\frac{\partial^{2+1}u}{\partial t^2\partial x}\cdot\frac{\partial^{2k-2}u}{\partial x^{2k-2}}\right) + (-1)^2\frac{\partial^{2+2}u}{\partial t^2\partial x^2}\cdot\frac{\partial^{2k-2}u}{\partial x^{2k-2}} =$$

$$= (-1)^0\frac{\partial}{\partial x}\left(\frac{\partial^{2+0}u}{\partial t^2}\cdot\frac{\partial^{2k-1}u}{\partial x^{2k-1}}\right) + (-1)^1\frac{\partial}{\partial x}\left(\frac{\partial^{2+1}u}{\partial t^2\partial x}\cdot\frac{\partial^{2k-2}u}{\partial x^{2k-2}}\right) + (-1)^2\frac{\partial^{2+2}u}{\partial t^2\partial x^2}\cdot\frac{\partial^{2k-3}u}{\partial x^{2k-3}} +$$

$$+ (-1)^3\frac{\partial^{2+3}u}{\partial t^2\partial x^3}\cdot\frac{\partial^{2k-3}u}{\partial x^{2k-3}} + \ldots + (-1)^{k-1}\frac{\partial}{\partial x}\left(\frac{\partial^{2+k-1}u}{\partial t^2\partial x^{k-1}}\cdot\frac{\partial^k u}{\partial x^k}\right) + (-1)^k\frac{\partial^{2+k}u}{\partial t^2\partial x^k}\cdot\frac{\partial^k u}{\partial x^k} =$$

$$= \sum_{m=0}^{k-1}(-1)^m\frac{\partial}{\partial x}\left(\frac{\partial^{2+m}u}{\partial t^2\partial x^m}\cdot\frac{\partial^{2k-m-1}u}{\partial x^{2k-m-1}}\right) + (-1)^k\frac{\partial}{\partial t}\left(\frac{\partial^{1+k}u}{\partial t\partial x^k}\cdot\frac{\partial^k u}{\partial x^k}\right) + (-1)^{k+1}\left(\frac{\partial^{k+1}u}{\partial t\partial x^k}\right)^2.$$

If m is odd, then $2k-m-1$ is even and according to (2) we have $\frac{\partial^{2k-m-1}u}{\partial x^{2k-m-1}} = 0$ at $x=0$ and $x=p$, in case of even m we have $\frac{\partial^{m+2}u}{\partial t\partial x^m} = 0$ at $x=0$ and $x=p$. Moreover $\frac{\partial^k u}{\partial x^k} = 0$ at $t=0$ and $t=T$. Consequently,

$$2\iint_\Omega \frac{\partial^2 u}{\partial t^2}\cdot\frac{\partial^{2k}u}{\partial x^{2k}}\ dxdt = -2\left\|\frac{\partial^{k+1}u}{\partial t\partial x^k}\right\|^2_{L_2(\Omega)}.$$

Substitutng it into (13) and dropping the coefficient 2 we get

$$\left\|\frac{\partial^{2k}u}{\partial x^{2k}}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^{k+1}u}{\partial t\partial x^k}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2(\Omega)} \le \|f\|^2_{L_2(\Omega)}. \qquad (14)$$

Adding (12) and (14) yields

$$\|u\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^{k+1} u}{\partial x^k \partial t}\right\|^2_{L_2(\Omega)} + \\ + \left\|\frac{\partial^{2k} u}{\partial x^{2k}}\right\|^2_{L_2(\Omega)} \le \left[(4T^2+1)^2 + 1 \right] \|f\|^2_{L_2(\Omega)}. \tag{15}$$

To obtain estimates for the norms of the form $\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}$, $m = 1,...,2k-1$ we use inequality

$$\left\|\frac{\partial^n u}{\partial x^n}\right\|^2_{L_2(\Omega)} \le \frac{1}{2}\left\|\frac{\partial^{n-1} u}{\partial x^{n-1}}\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\frac{\partial^{n+1} u}{\partial x^{n+1}}\right\|^2_{L_2(\Omega)}. \tag{16}$$

that can easily be checked. If we sum inequalities (16) over n from 1 to $2k-1$ and use (15), we get

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^{2k-1} u}{\partial x^{2k-1}}\right\|^2_{L_2(\Omega)} \le \left[(4T^2+1)^2 + 1 \right] \|f\|^2_{L_2(\Omega)}. \tag{17_1}$$

Now summing up inequalities (16) over n from 2 to $2k-2$ according to (17_1) we have

$$\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^{2k-2} u}{\partial x^{2k-2}}\right\|^2_{L_2(\Omega)} \le \left[(4T^2+1)^2 + 1 \right] \|f\|^2_{L_2(\Omega)} \tag{17_2}$$

Proceeding in this way we obtain

$$\left\|\frac{\partial^3 u}{\partial x^3}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^{2k-3} u}{\partial x^{2k-3}}\right\|^2_{L_2(\Omega)} \le \left[(4T^2+1)^2 + 1 \right] \|f\|^2_{L_2(\Omega)}. \tag{17_3}$$

....

$$\left\|\frac{\partial^{k-1}u}{\partial x^{k-1}}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{k+1}u}{\partial x^{k+1}}\right\|^2_{L_2(\Omega)}\le\left[(4T^2+1)^2+1\right]\,\|f\|^2_{L_2(\Omega)}. \qquad (17_{\text{k-1}})$$

Adding inequalities (17_1), (17_2),..., $(17_{\text{k-1}})$ yields

$$\sum_{\substack{m=1\\ m\neq k}}^{2k-1}\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}\le(k-1)\left[(4T^2+1)^2+1\right]\,\|f\|^2_{L_2(\Omega)}. \qquad (18)$$

Adding inequalities (15) and (18) we obtain

$$\sum_{m=0}^{2}\left\|\frac{\partial^m u}{\partial t^m}\right\|^2_{L_2(\Omega)}+\sum_{m=1}^{2k}\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{k+1}u}{\partial x^k\partial t}\right\|^2_{L_2(\Omega)}\le k\left[(4T^2+1)^2+1\right]\,\|f\|^2_{L_2(\Omega)}. \qquad (19)$$

Summing up the inequalities

$$\left\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\right\|^2_{L_2(\Omega)}\le\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{2m-2}u}{\partial x^{2m-2}}\right\|^2_{L_2(\Omega)},$$

which proof is evident, over m from 2 to k according to (19) we have

$$\sum_{m=2}^{k}\left\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\right\|^2_{L_2(\Omega)}\le k(k-1)\left[(4T^2+1)^2+1\right]\,\|f\|^2_{L_2(\Omega)}. \qquad (20)$$

Adding (19) and (20) we get

$$\sum_{m=0}^{2}\left\|\frac{\partial^m u}{\partial t^m}\right\|^2_{L_2(\Omega)}+\sum_{m=2}^{k+1}\left\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\right\|^2_{L_2(\Omega)}+\sum_{m=1}^{2k}\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}\le k^2\left[(4T^2+1)^2+1\right]\,\|f\|^2_{L_2(\Omega)}$$

or

$$\|u\|^2_{W_2^{2k,2}(\Omega)}\le C^2\|f\|^2_{L_2(\Omega)}, \qquad (21)$$

where $C^2\le k^2\left[\left(4T^2+1\right)^2+1\right]$.

This proves Lemma 1.

The Regular Solvability of the Problem 1

It is true the following

Theorem 1. Let $f(x,t) \in W_1(\Omega)$ if k is even and $f(x,t) \in W_2(\Omega)$ if k is odd and numbers P and T satisfy the condition

$$\left| sin\left(\frac{n\pi}{p}\right)^k T \right| \geq \delta > 0, \ \forall n \in N . \tag{22}$$

Then there exists a regular solution of Problem 1.

We search a regular solution of Problem 1 in the form of Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{23}$$

expanded in full orthonormal system

$$X_n(x) = \sqrt{\frac{2}{p}} sin \lambda_n x, \ \lambda_n = \frac{n\pi}{p}, n \in N ,$$

in $L_2(0,p)$.

It is clear that $u(x,t)$ satisfies conditions (2). We expand the function $f(x,t)$ into the Fourier series in functions $X_n(x)$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x), \tag{24}$$

where

$$f_n(t) = \int_0^p f(x,t) X_n(x) \ dx . \tag{25}$$

Substituting (23) and (24) into the equation (1) we obtain the following equation

$$u_n''(t) - (-1)^k \lambda_n^{2k} u_n(t) = -f_n(t). \tag{26}$$

for unknown function $u_n(t)$. Conditions (3) take the form

$$u_n(0)=0,\ \ u_n(T)=0. \tag{27}$$

The solution of the equation (26) satisfying conditions (27) has the form

$$u(x,t)=\sum_{n=1}^{\infty} X_n(x)\cdot\frac{1}{\lambda_n^k}\int_0^T K_n^{(1)}(t,\tau)f_n(\tau)d\tau, \tag{28}$$

if k is even, and has the form

$$u(x,t)=\sum_{n=1}^{\infty} X_n(x)\cdot\frac{1}{\lambda_n^k}\int_0^T K_n^{(2)}(t,\tau)f_n(\tau)d\tau, \tag{29}$$

if k is odd, where

$$K_n^{(1)}(t,\tau)=\begin{cases}\dfrac{sh\lambda_n^k\tau\cdot sh\lambda_n^k(T-t)}{sh\lambda_n^kT},\ 0\le\tau\le t,\\[2ex] \dfrac{sh\lambda_n^kt\cdot sh\lambda_n^k(T-\tau)}{sh\lambda_n^kT},\ t\le\tau\le T,\end{cases}$$

$$K_n^{(2)}(t,\tau)=\begin{cases}\dfrac{sin\,\lambda_n^k\tau\cdot sin\,\lambda_n^k(T-t)}{sin\,\lambda_n^kT},\ 0\le\tau\le t,\\[2ex] \dfrac{sin\,\lambda_n^kt\cdot sin\,\lambda_n^k(T-\tau)}{sin\,\lambda_n^kT},\ t\le\tau\le T,\end{cases}$$

with

$$K_n^{(i)}(t,\tau)=K_n^{(i)}(\tau,t),\quad i=1,2\ ,$$

$$K_n^{(1)}(t,\tau)\le\frac{C_0}{e^{\lambda_n^k|t-\tau|}},\ C_0=const>0, \tag{30}$$

$$\left|K_n^{(2)}(t,\tau)\right|\le\frac{1}{\delta}. \tag{31}$$

Let k be an even number. We have to prove uniformly convergence of the series (28) and

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \sum_{n=1}^{\infty} (-1)^k \lambda_n^{2k} \cdot \frac{1}{\lambda_n^k} X_n(x) \int_0^T K_n^{(1)}(t,\tau) f_n(\tau) d\tau, \tag{32}$$

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{n=1}^{\infty} X_n(x) f_n(t) + \sum_{n=1}^{\infty} \lambda_n^{2k} \frac{1}{\lambda_n^k} X_n(x) \int_0^T K_n^{(1)}(t,\tau) f_n(\tau) d\tau, \tag{33}$$

If we show uniformly convergence of the series

$$\sum_{n=1}^{\infty} \lambda_n^k \cdot X_n(x) \int_0^T K_n^{(1)}(t,\tau) f_n(\tau) d\tau, \tag{34}$$

then which implies uniformly convergence of the series (28), (32), (33).

In the equality (25) we integrate the integral

$$f_n(t) = \frac{1}{\lambda_n} \overline{f_n}(t)$$

by parts, where

$$\overline{f_n}(t) = \int_0^p \frac{\partial f}{\partial x} \sqrt{\frac{2}{p}} \cos \lambda_n x dx .$$

Since $\frac{\partial f}{\partial x} \in Lip_\alpha [0, p]$ is uniformly with respect to t, then [15]

$$\left| \overline{f_n}(t) \right| \le \frac{C_1}{\lambda_n^\alpha}, \quad C_0 = const > 0, \ 0 < \alpha < 1 .$$

So

$$\left| f_n(\tau) \right| \le \frac{C_1}{\lambda_n^{1+\alpha}} . \tag{35}$$

We next turn to estimating the integral in (34). According to (30) and (35) we have

$$\left|\int_0^T K_n^{(1)}(t,\tau) f_n(\tau)\,d\tau\right| \le \int_0^T K_n^{(1)}(t,\tau)\left|f_n(\tau)\right| d\tau \le$$
$$\le \frac{C_1 C_0}{\lambda_n^{1+\alpha}} \int_0^T \frac{d\tau}{e^{\lambda_n^k |t-\tau|}} = \frac{C_1 C_0}{\lambda_n^{1+\alpha}}\left[\int_o^t e^{-\lambda_n^k (t-\tau)} d\tau + \int_t^T e^{-\lambda_n^k (\tau - t)} d\tau\right] = \tag{36}$$
$$= \frac{C_1 C_0}{\lambda_n^{1+\alpha}}\left[\frac{1}{\lambda_n^k}\left(1 - e^{-\lambda_n^k t}\right) - \frac{1}{\lambda_n^k}\left(e^{-\lambda_n^k (T-t)} - 1\right)\right] \le \frac{2C_1 C_0}{\lambda_n^{1+\alpha}} \cdot \frac{1}{\lambda_n^k}.$$

The estimate (36) implies uniformly convergence of the series (34), (33), (32), (28).

This finishes the proof of Theorem 1 for even k .

We now turn to the case where k is odd. It has to be shown uniformly convergence of the series (29) and

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{n=1}^{\infty} f_n(t) X_n(x) - \sum_{n=1}^{\infty} \lambda_n^k X_n(x) \int_0^T K_n^{(2)}(t,\tau) f_n(\tau) d\tau, \tag{37}$$

$$\frac{\partial^2 u}{\partial x^{2k}} = -\sum_{n=1}^{\infty} \lambda_n^k X_n(x) \int_0^T K_n^{(2)}(t,\tau) f_n(\tau) d\tau, \tag{38}$$

It suffices to show convergence of the series (38).

Let $f \in W_2(\Omega)$. We integrate the integral (25) by parts $k+1$ times

$$f_n(t) = -\frac{1}{\lambda_n^{k+1}} \overline{f_n}(t), \tag{39}$$

where $\overline{f_n}(t) = \int_0^p \frac{\partial^{k+1} f}{\partial x^{k+1}} X_n(x)\,dx$.

We proceed to estimate the integral. According to (31) and (39) we obtain

$$\left|\int_0^T K_n^{(2)}(t,\tau) f_n(\tau)d\tau\right| \le \int_0^T \left|K_n^{(2)}(t,\tau)\right| \; \left|f_n(\tau)\right|d\tau \le \frac{1}{\delta\lambda_n^{k+1}}\int_0^T \left|f_n(\tau)\right|d\tau \le$$
$$\le \frac{1}{\delta\lambda_n^{k+1}}\sqrt{\int_0^T d\tau \cdot \int_0^T \left|f_n(\tau)\right|^2 d\tau} = \frac{\sqrt{T}}{\delta}\cdot\frac{1}{\lambda_n^{k+1}}\left\|\overline{f_n}\right\|_{L_2(0,T)} \tag{40}$$

Here we have used the Cauchy-Schwartz inequality. Taking into account (40) yields

$$\sum_{n=1}^{\infty}\lambda_n^k\left|X_n(x)\int_0^T K_n^{(2)}(t,\tau) f_n(\tau)d\tau\right| \le \frac{1}{\delta}\sqrt{\frac{2T}{p}}\sum_{n=1}^{\infty}\lambda_n^k\cdot\frac{1}{\lambda_n^{k+1}}\left\|\overline{f_n}\right\|_{L_2(0,T)} =$$
$$= \frac{1}{\delta}\sqrt{\frac{2T}{p}}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\cdot\left\|\overline{f_n}\right\|_{L_2(0,T)} \le \frac{1}{2\delta}\sqrt{\frac{2T}{p}}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}+\sum_{n=1}^{\infty}\left\|\overline{f_n}\right\|_{L_2(0,T)}^2\right) < \infty,$$

As

$$\sum_{n=1}^{\infty}\left\|\overline{f_n}\right\|_{L_2(0,T)}^2 = \left\|\frac{\partial^{k+1} f}{\partial x^{2k+1}}\right\|_{L_2(\Omega)}^2,$$

then the series (38) converges uniformly.

By the estimate (40) the series (29) and (37) are also convergent uniformly, and the proof of Theorem 1 is completed.

Remark. As to the condition (3)

$$u(x,0) = u(x,T) = 0, \qquad 0 \le x \le p\ , \tag{3}$$

the condition is necessary at *t=T*. If we don't impose any condition at *t=T* and change it to

$$u_t\big|_{t=0} = 0,$$

then the problem is not correct for even *k*.

Indeed, if this is the case, then we have the following equation

$$u_n''(t)-\lambda_n^{2k}u_n(t)=-f_n(t)$$

for $u_n(t)$. The general solution of this equation has the form

$$u_n(t)=a_n(0)e^{\lambda_n^k t}+b_n(0)e^{-\lambda_n^k t}-\frac{1}{\lambda_n^k}\int_0^t f_n(\tau)sh\lambda_n^k(t-\tau)d\tau$$

We require the obtained solution to satisfy the following conditions

$$u_n(0)=0,\ u_n'(0)=0 .$$

Then we get

$$u_n(0)=a_n(0)+b_n(0)=0,\quad b_n(0)=-a_n(0)$$
$$u_n'(0)=\lambda_n^k a_n(0)-\lambda_n^k b_n(0)=0\Rightarrow 2\lambda_n^k a_n(0)=0\Rightarrow a_n(0)=0,\ b_n(0)=0.$$

It is clear that the sequence

$$u_n(t)=-\frac{1}{\lambda_n^k}\int_0^t f_n(t)sh\lambda_n^k(t-\tau)d\tau$$

doesn't converge. Thus the problem is incorrect.

Lemma 2. Let k is odd number. Then the solution (29) satisfies the estimate

$$\|u\|_{W_2^{k,1}(\Omega)}\le C_2\|f\|_{L_2(\Omega)}, \tag{41}$$

where C_2 positive constant depending only on sizes of the domain and not depending on the function $u(x,t)$.

Proof. We rewrite the solution (29) in the form

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x), \tag{42}$$

where

$$u_n(t) = \frac{1}{\lambda_n^k} \int_0^T K_n^{(2)}(t,\tau) f_n(\tau) d\tau \tag{43}$$

We evaluate the norm $\|u_n\|$. By (31) and Cauchy-Schwartz inequality we get

$$|u_n(t)|^2 = \frac{1}{\lambda_n^{2k}} \left| \int_0^T K_n^{(2)}(t,\tau) f_n(\tau) d\tau \right|^2 \le$$
$$\le \frac{1}{\lambda_n^{2k}} \int_0^T \left| K_n^{(2)}(t,\tau) \right|^2 d\tau \int_0^T |f_n(\tau)|^2 d\tau \le \frac{T}{\delta^2 \lambda_n^{2k}} \|f_n\|^2_{L_2(0,T)}.$$

Integrating the inequality

$$|u_n(t)|^2 \le \frac{T}{\delta^2 \lambda_n^{2k}} \|f_n\|^2_{L_2(0,T)}$$

with respect to t from 0 to T we obtain

$$\|u_n\|^2_{L_2(0,T)} \le \frac{T^2}{\delta^2 \lambda_n^{2k}} \|f_n\|^2_{L_2(0,T)}. \tag{44}$$

By using (44) we estimate $\|u\|_{L_2(\Omega)}$.

$$\|u\|^2_{L_2(\Omega)} = \left(\sum_{n=1}^{\infty} u_n(t) X_n(x), \sum_{m=1}^{\infty} u_n(t) X_m(x) \right)_{L_2(\Omega)} =$$
$$= \int_0^T \int_0^p \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_n(t) u_m(t) X_n(x) X_m(x) dx dt =$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^T u_n(t) u_m(t) dt \int_0^p X_n(x) X_m(x) dx = \sum_{m=1}^{\infty} \int_0^T u_n^2(t) dt =$$
$$= \sum_{n=1}^{\infty} \|u_n\|^2_{L_2(0,T)} \le \frac{T^2 p^{2k}}{\delta^2 \pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \|f_n\|^2_{L_2(0,T)} \le$$
$$\le \frac{T^2 p^2}{\delta^2 \pi^2} \sum_{n=1}^{\infty} \|f_n\|^2_{L_2(0,T)} = \frac{T^{2k} p^{2k}}{\delta^2 \pi^{2k}} \|f\|^2_{L_2(\Omega)}.$$

So

$$\|u\|_{L_2(\Omega)}^2 \le \frac{T^2 p^2}{\delta^2 \pi^2}\|f\|_{L_2(\Omega)}. \tag{45}$$

Now we estimate the norm $\|u_t\|_{L_2[\Omega]}$. To this end we first estimate $\left|u_n^{'}\right|_{L_2(\Omega)}$.

$$u_n^{'}(t) = -\int_0^t \frac{\sin\lambda_n^k\tau\cdot\cos\lambda_n^k(T-t)}{\sin\lambda_n^k T} f_n(\tau)\,d\tau +$$

$$+\int_t^T \frac{\cos\lambda_n^k t\cdot\sin\lambda_n^k(T-\tau)}{\sin\lambda_n^k T} f_n(\tau)\,d\tau.$$

$$\left|u_n^{'}(t)\right| \le \frac{1}{\delta}\int_0^t |f_n(\tau)|d\tau + \frac{1}{\delta}\int_t^T |f_n(\tau)|d\tau = \frac{1}{\delta}\int_0^T |f_n(\tau)|d\tau \le$$

$$\le \frac{1}{\delta}\sqrt{\int_0^T 1^2 d\tau}\cdot\sqrt{\int_0^T |f_n(\tau)|^2 d\tau} = \frac{\sqrt{T}}{\delta}\|f_n\|_{L_2(0,T)}.$$

Squaring this inequality and integrating with respect to t from 0 to T we obtain

$$\left\|u_n^{'}\right\|_{L_2(0,T)}^2 \le \frac{T^2}{\delta^2}\|f_n\|_{L_2(0,T)}^2.$$

Using this inequality and the Parceval identity yields

$$\|u_t\|_{L_2(\Omega)}^2 = \left(\sum_{n=1}^{\infty} u_n^{'}(t)X_n(x), \sum_{m=1}^{\infty} u_m^{'}(t)X_m(x)\right) =$$

$$= \sum_{n=1}^{\infty}\left\|u_n^{'}\right\|_{L_2(0,T)}^2 \le \frac{T^2}{\delta^2}\sum_{n=1}^{\infty}\|f_n\|_{L_2(\Omega)}^2 = \frac{T^2}{\delta^2}\|f\|_{L_2(\Omega)}^2.$$

From here we get

$$\|u_t\|_{L_2(\Omega)}^2 \le \frac{T^2}{\delta^2}\|f\|_{L_2(\Omega)}^2. \tag{46}$$

We estimate $\|u_x\|_{L_2(\Omega)}$. Combining (44) and the Bessel inequality gives

$$\|u_x\|^2_{L_2(\Omega)} = \left(\sum_{n=1}^{\infty} u_n(t) X_n^{'}(x), \sum_{m=1}^{\infty} u_m(t) X_m^{'}(x) \right) =$$
$$= \sum_{n=1}^{\infty} \lambda_n^2 \|u_n^{'}\|^2_{L_2(0,T)} \le \frac{T^2}{\delta^2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2k-2}} \|f_n\|^2_{L_2(0,T)} =$$
$$= \frac{T^2}{\delta^2} \cdot \frac{p^{2k-2}}{\pi^{2k-2}} \sum_{n=1}^{\infty} \frac{1}{n^{2k-2}} \|f_n\|^2_{L_2(0,T)} \le \left(\frac{T \cdot p^{k-1}}{\delta \pi^{k-1}} \right)^2 \sum_{n=1}^{\infty} \|f_n\|^2_{L_2(0,T)};$$

$$\|u_x\|^2_{L_2(\Omega)} \le \left(\frac{T \cdot p^{k-1}}{\delta \pi^{k-1}} \right)^2 \|f\|^2_{L_2(\Omega)}. \qquad (47_1)$$

For $\left\| \frac{\partial^2 u}{\partial x^2} \right\|^2_{L_2(\Omega)}$ we have the following estimation

$$\left\| \frac{\partial^2 u}{\partial x^2} \right\|^2_{L_2(\Omega)} \le \left(\frac{T \cdot p^{k-1}}{\delta \pi^{k-1}} \right)^2 \|f\|^2_{L_2(\Omega)}. \qquad (47_2)$$

....

$$\left\| \frac{\partial^k u}{\partial x^k} \right\|^2_{L_2(\Omega)} \le \frac{T^2}{\delta^2} \|f\|^2_{L_2(\Omega)}. \qquad (47_k)$$

Adding the inequalities (45), (46), (47_1), ..., (47_k) yields

$$\sum_{m=0}^{k} \left\| \frac{\partial^m u}{\partial x^m} \right\|^2_{L_2(\Omega)} + \left\| \frac{\partial u}{\partial t} \right\|^2_{L_2(\Omega)} \le C_2^2 \|f\|^2_{L_2(\Omega)}$$

or

$$\|u\|_{W_2^{k,1}(\Omega)} \le C_2 \|f\|_{L_2(\Omega)},$$

where $C_2 = C_2(p, T, \delta, \pi, k) = const > 0$.

The proof of Lemma 2 is completed.

The Strong Solvability

It is true the following

Theorem 2. For any $f \in L_2(\Omega)$ there exists a unique strong solution of Problem 1 and it satisfies estimation (7), if *k* is even, and estimation (41) if *k* is odd.

Proof. Let f be an arbitrary function in $L_2(\Omega)$ and *k* be an even number. According to the fact that $W_1(\Omega)$ is dense in $L_2(\Omega)$ there exists a sequence $\{f_n\} \subset W_1(\Omega)$, $n \in N$ such that $\|f_n - f\|_{L_2(\Omega)} \to 0$ as $n \to \infty$ .Consequently, $\{f_n\}$ is Cauchy sequence in $L_2(\Omega)$. We denote by $u_n(x,t) \in V(\Omega)$ the solution of the equation (1) with the right part $f_n(x,t)$. By (7) we have

$$\|u_n - u_m\|_{W_2^{2k,2}(\Omega)} \le C\|f_n - f_m\|_{L_2(\Omega)} \to 0, \; n,m \to \infty, \tag{48}$$

that is $\{u_n\}$ is a Cauchy sequence in $\overset{0}{W}{}_2^{2k,2}(\Omega)$. According to completeness of the space $\overset{0}{W}{}_2^{2k,2}(\Omega)$ there exists a unique limit $u(x,t) = \lim\limits_{n\to\infty} u_n(x,t) \in \overset{0}{W}{}_2^{2k,2}(\Omega)$ which is the strong solution of Problem 1. Passing to limit in inequality $\|u_n\|_{W_2^{2k,2}(\Omega)} \le C\|f_n\|_{L_2(\Omega)}$ as $n \to \infty$ we conclude that estimation (7) is also true for the strong solution $u(x,t)$ Passing to limit in equation $Lu_n = f_n$, $u_n \in V(\Omega)$, $f_n \in W_1(\Omega)$, as $n \to \infty$ we get $Lu_n = f_n$, $u_n \in \overset{0}{W}{}_2^{2k,2}(\Omega)$, $f \in L_2(\Omega)$. Consequently, the strong solution is a solution almost everywhere. In a similar way one can prove that Problem 1 is strong solvable in the space $\overset{0}{W}{}_2^{k,1}(\Omega)$ in case of odd *k*.

Spectrum of Problem 1

The spectrum of a problem is the set of eigenvalues of the operator of the problem. We examine spectrum of the problem in case of even *k*. The investigation of the spectrum for odd *k* is similar.

We rewrite the solution (28) as

$$u(x,t)=\int_0^p\int_0^T K^{(1)}(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau\,, \tag{49}$$

where

$$K^{(1)}(x,t;\xi,\tau)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)}{\lambda_n^k}K_n^{(1)}(t,\tau)\,. \tag{50}$$

As $K_n^{(1)}(t,\tau)$ is symmetric, then $K^{(1)}(x,t;\xi,\tau)$ is symmetric. The estimation (30) implies its boundedness, i.e.

$$\left|K_n^{(1)}(x,t;\xi,\tau)\right|\le C_2 \tag{51}$$

Combining (49) with (51) we conclude that it is defined bounded symmetric operator L^{-1} on $W_1(\Omega)$ which is inverse of the operator L and acts from $W_1(\Omega)$ to $V(\Omega)$ by the rule

$$\left(L^{-1}f\right)(x,t)=\int_0^p\int_0^T K^{(1)}(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau\,, \tag{52}$$

It can be extended to whole space $L_2(\Omega)$. This extension, we denote it by $\overline{L^{-1}}$, is the closure of L^{-1}, $D(\overline{L^{-1}})=L_2(\Omega)$. The operator $\overline{L^{-1}}$ is symmetric, bounded, and defined on the whole space $L_2(\Omega)$, so it is self-adjoint. It follows from (51) that $K^{(1)}(x,t;\xi,\tau)\in L_2(\Omega\times\Omega)$ therefore $\overline{L^{-1}}$ is a compact operator in $L_2(\Omega)$. Then the spectrum of the operator $\overline{L^{-1}}$ is discrete and consists of real eigenvalues of finite multiplicity. The relation between eigenvalues of the operators $\overline{L^{-1}}$ and $\overline{L}$ is as follows (Dezin,1980): if $\mu_n\ne 0$ is an eigenvalue of the operator $\overline{L^{-1}}$, then μ^{-1} is eigenvalue of the operator $\overline{L}$.

Thus, in case of even *k* the spectrum of Problem 1 consists of real eigenvalues of finite multiplicity.

A similar assertion is also true in case of odd k.

Corollary. Problem 1 is self adjoint for all k.

CONCLUSION

In this article we have investigated four boundary value problems for the equation of the even order in a rectangular domain. One of these problems is studied in detail. Other problems can be handled in much the same way. In case even k we have obtained a priori estimate for the solution in the norm of the space $W_2^{2k,2}(\Omega)$, proved its regular and strong solvability almost everywhere. In case of odd k we have driven the estimate for the regular solution in the norm of the space $W_2^{k,1}(\Omega)$. The spectrum of the problem has been researched and its discreteness has been proved. The self-adjointness of problem has been established.

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